2015-2016 CATL Teaching and Learning Grant Report
Lesson Study

Due June 1, 2016. Email a single Word or PDF file to catl@uwlax.edu
Use the following format to prepare your completed report. The report will be *published* on the College Lesson Study Blog.

PART I: BACKGROUND	
Title	Sequences and Series
Authors	Dr. Whitney George (UW-La Crosse) and Dr. Nathan Warnberg (UW-La Crosse)
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Discipline(s)	Mathematics
Submission Date	June 1, 2016
Course Name	Calculus II
Course Description	A continuation of Calculus I with a rigorous introduction to sequences and series. Topics include techniques of integration and indeterminate forms, improper integrals, applications of integrals to the physical sciences, tests for the convergence of series, absolute convergence, power series, and Taylor's Theorem with Remainder. First order linear differential equations are explored, as well as the geometry of space.
Abstract	This lesson study is a continuation of our 2015 Lesson Study on Sequences and Series. In the 2015 lesson study, our learning goal was to relate sequences and series. We realized that this was too large of a goal and for our 2016 lesson study, we narrowed our learning objectives to one objective: Have students relate the sequence of partial sums to a given series. We simplified the lesson study to major concepts and left out many of the details including notation that can often distract students from the major ideas. We also asked repetitive questions for different series that were progressively harder. We found that the students had a better understanding of series and the underlying defining sequence. However students were not able to make the connection between the convergence of the sequence of partial sums and the convergence of the series as well as we had hoped. These were still two separate ideas for many of the students.
PART II: THE LESSON	
Learning Goals	The learning goals for the students are the following: 1. Understand the difference between a series which converges or diverges. 2. Understand the relationship between the defining sequence of a series and the actual series. 3. Understand the relationship between the series the sequence of partial sums. 4. Understand how the convergence or divergence of the sequence of partial sums is related to the convergence or divergence of the corresponding series.
Lesson Plan	Day 1: The lesson study started with motivation for why we are studying series. We find that students are often confused by the material as series can seem completely unrelated to the rest of the Calculus II material even though series is at the heart of the material. Once the students had an understanding for why series are important, we tied in the material to past problems including how to integrate functions without an elementary antiderivative. Next, we introduced series of

	numbers and discussed what it means for an infinite sum to converge or diverge. We used graphs to help build intuition for the students. Again, we brought in past knowledge such as partial fraction decompositions and polynomial division (both techniques used in integration earlier in the semester) to help motivate and explain telescoping series and geometric series. We ended class with introducing only the essential definitions leaving all other definitions for later to help build a solid foundation early on. The students were given a worksheet to complete at home before the next day's class. Day 2: We broke the students up into small groups and allowed them to discuss their answers and questions on the worksheet from the day before. The observers walked around and took notes on what were observed the students saying and doing. After 20 minutes, we worked on the worksheet together as a class and addressed any major questions or concerns of the students.
PART III: THE STUDY	
Approach	Dr. Whitney George led the class lecture on the first day of the lesson study. Dr. Nathan Warnberg and Dr. Josh Hertel sat in the class near students and observed the lesson as well as the students. Students were given a worksheet at the end of class and asked to complete it for the next day. On day 2, students were split into small groups to discuss their results. All three observers walked around observing students and asking clarifying questions about their work when necessary. After 20 minutes, Dr. Whitney George led the class discussion on the solutions to the worksheet.
Findings/Discussion	The Lesson Study was a continuation on a lesson study completed in 2015. We learned many valuable lessons from the previous year and felt that we could improve the lesson which became the 2016 lesson study. One of the changes that we made was to limit the notation, theorems, and definitions unless they were absolutely necessary. We felt that students easily get bogged down with too much at one time and we wanted to develop a strong foundation for the students. This was a success. The students were also able to find the sequence of partial sums given a given series. Students were even able to find the convergence/divergence of the sequence of partial sums. Despite this, we found that the students lacked understanding that the convergence of the sequence of partial sums determined the convergence of the series. There seemed to be a disconnect between these two concepts even though the students were able to find the convergence/divergence of both the series and sequence of partial sums.
References	N/A
APPENDIX	
Dissemination	We will be presenting our lesson study and findings at the 2016 UWL Conference on Teaching and Learning. We also plan on publishing our experiences with lesson studies in the near future.

Outline of Lesson Study for 2016

1 Motivation for material

Start with the two functions:

$$f(x) = \frac{e^x}{x} \qquad \text{and} \qquad g(x) = \frac{1}{x} + \frac{1}{1} + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720}$$

1.1 Questions

1. What are some difference between these two functions?
2. What are some similarities between these two functions?

Estimated Time: 3 minutes

GRAPH THESE FUNCTIONS IN DESMOS: Lesson Study 1

3. Is there a pattern that can be seen in the terms of $g(x)$?

IF THE STUDENTS SEE PATTERN, ADD MORE TERMS TO $g(x)$ IN DESMOS AND MAKE OBSERVATION ABOUT THE APPROXIMATION

We see that $\frac{e^x}{x} \approx \frac{1}{x} + \frac{1}{1} + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720}$. Recall that we have seen the integral $\int \frac{e^x}{x}\,dx$. So, if we have this approximation, then we could write down

$$\int \frac{e^x}{x}\,dx = \int \frac{1}{x} + \frac{1}{1} + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720}\,dx$$

Estimated Time: 10 minutes

and evaluate this (easy) integral.

PLUG INTO WOLFRAM ALPHA TO SHOW THEM THE DIFFERENCE BETWEEN THESE

`http://www.wolframalpha.com/input/?i=int_1%5E3+e%5Ex%2Fxdx`

`http://www.wolframalpha.com/input/?i=int_1%5E3+(1%2Fx%2B1%2Bx%2F2%2Bx%5E2%2F6%2Bx%5E3%2F24%2Bx%5E4%2F120%2Bx%5E5%2F720)dx`

TAKE AWAYS

- Every function can be written as an (infinite) polynomial (on some interval)
- Need to understand infinite polynomials
- To do this, we need to understand infinite sums.

2 Introduction to Infinite Sums

1. Can you sum up infinitely many things and get a finite total?

   - $1 + 1 + 1 + 1 + \dots$ (INFINITE)
   - $1 - 1 + 1 - 1 + \dots$ (UNDEFINED)
   - $1 + \frac{1}{2^1} + \frac{1}{3^1} + \frac{1}{4^1} + \frac{1}{5^1} + \dots$ (INFINITE)
   - $1 + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \dots$ (FINITE-p-SERIES)
   - $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ (FINITE-GEOMETRIC)

PUT THESE INTO DESMOS AND TRY TO LEAD THEM TO THE FACT THAT IF $\lim_{n\to\infty} a_n \neq 0$, THEN THE SUM CANNOT BE FINITE: LESSON STUDY 2

USE THE BOX TO SHOW THAT THE LAST SUM GOES TO 1

Estimated Time: 10 minutes

<u>**TAKE AWAYS**</u>

- Infinite sums involve a lot of subtle ideas
- Intuition can quickly fail you
- If $\lim_{n\to\infty} a_n \neq 0$, then $a_1 + a_2 + a_3 + a_4 + \dots$ cannot be finite.
- On the other hand, just because $\lim_{n\to\infty} a_n = 0$, this does not guarantee that $a_1+a_2+a_3+a_4+\dots$ is finite.

3 Mathematical Methods

Mention how certain mathematical concepts can be applied in different settings:

1. Determine if $\int_1^\infty \frac{1}{x^{1.1}}\,dx$ converges or diverges.

TIE THIS INTO KNOWING THAT $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ CONVERGES BY COMPARING AREA-DESMOS LESSON STUDY 3

2. Integrate $\int \frac{2}{x(x+2)}\,dx$ (PARTIAL FRACTION DECOMPOSITION)

3. Use polynomial division to find $\frac{1-x^9}{1-x}$. Then, generalize to $\frac{1-x^{n+1}}{1-x}$

Estimated Time: 10 minutes

4 Infinite Sums

Put the next two examples on the board and ask the following questions. Put them into Desmos, too.

1. Need to find a pattern for s_n
2. What is the connection between the sequence a_n and the sequence s_n?
3. What is a good name for s_n? (SEQUENCE OF PARTIAL SUMS)
4. What is $\lim\limits_{n\to\infty} a_n$? (BRING UP TERMINOLOGY OF CONVERGENCE)
5. What is $\lim\limits_{n\to\infty} s_n$?
6. Does is make sense for $\lim\limits_{n\to\infty} s_n$ to represent the infinite sum?

Estimated Time: 20 minutes

4.1 Investigate the series $\sum_{1}^{\infty} \frac{1}{2^n}$

a_1	$\frac{1}{2^1}$
a_2	$\frac{1}{2^2}$
a_3	$\frac{1}{2^3}$
a_4	$\frac{1}{2^4}$
a_5	$\frac{1}{2^5}$
a_6	$\frac{1}{2^6}$

s_1	$\frac{1}{2^1}$
	$= \frac{1-\frac{1}{2}^2}{1-\frac{1}{2}} - 1$
s_2	$\frac{1}{2^1} + \frac{1}{2^2}$
	$= \frac{1-\frac{1}{2}^3}{1-\frac{1}{2}} - 1$
s_3	$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}$
	$= \frac{1-\frac{1}{2}^4}{1-\frac{1}{2}} - 1$
s_4	$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$
	$= \frac{1-\frac{1}{2}^5}{1-\frac{1}{2}} - 1$
s_5	$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}$
	$= \frac{1-\frac{1}{2}^6}{1-\frac{1}{2}} - 1$
s_6	$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$
	$= \frac{1-\frac{1}{2}^7}{1-\frac{1}{2}} - 1$

4.2 Investigate the series $\sum_{1}^{\infty} \frac{2}{n(n+2)}$

a_1	$\frac{2}{1\cdot 3} = \frac{1}{1} - \frac{1}{3}$
a_2	$\frac{2}{2\cdot 4} = \frac{1}{2} - \frac{1}{4}$
a_3	$\frac{2}{3\cdot 5} = \frac{1}{3} - \frac{1}{5}$
a_4	$\frac{2}{4\cdot 6} = \frac{1}{4} - \frac{1}{6}$
a_5	$\frac{2}{5\cdot 7} = \frac{1}{5} - \frac{1}{7}$
a_6	$\frac{2}{6\cdot 8} = \frac{1}{6} - \frac{1}{8}$

s_1	$\frac{1}{1} - \frac{1}{3}$
	$= \frac{1}{1} - \frac{1}{3}$
s_2	$\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$
	$= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$
s_3	$\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$
	$= \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$
s_4	$\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$
	$= \frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$
s_5	$\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \frac{1}{7}$
	$= \frac{1}{1} + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$
s_6	$\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \frac{1}{7} + \cancel{\frac{1}{6}} - \frac{1}{8}$
	$= \frac{1}{1} + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}$

TAKE AWAYS

- $s_n = a_1 + a_2 + a_3 + \cdots + a_n$
- $\lim_{n\to\infty} s_n = a_1 + a_2 + a_3 + a_4 + \dots$
- There are multiple sequences involved....need to keep them straight

Outline of Lesson Study for 2016 for Observers

1 Motivation for material

Start with the two functions:

$$f(x) = \frac{e^x}{x} \qquad \text{and} \qquad g(x) = \frac{1}{x} + \frac{1}{1} + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720}$$

1. How did the lesson relate $f(x)$ and $g(x)$?

2. How was it presented that this idea can be extended to other functions?

3. Was the importance of infinite summing emphasized enough?

Estimated Time: 13 minutes

2 Introduction to Infinite Sums

Can you sum up infinitely many things and get a finite total?

- $1 + 1 + 1 + 1 + \dots$ (INFINITE)
- $1 - 1 + 1 - 1 + \dots$ (UNDEFINED)
- $1 + \frac{1}{2^1} + \frac{1}{3^1} + \frac{1}{4^1} + \frac{1}{5^1} + \dots$ (INFINITE)
- $1 + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \dots$ (FINITE-p-SERIES)
- $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ (FINITE-GEOMETRIC)

1. How were the subtleties observed?

2. Was it made clear with the examples that small differences in the series will result in vastly different conclusions about convergence?

3. Was it clear that if a_n doesn't converge to 0, then s_n doesn't converge to zero?

Estimated Time: 10 minutes

3 Mathematical Methods

Mention how certain mathematical concepts can be applied in different settings:

- Integrate $\int \frac{2}{x(x+2)}\,dx$ (PARTIAL FRACTION DECOMPOSITION)
- Use polynomial division to find $\frac{1-x^9}{1-x}$. Then, generalize to $\frac{1-x^{n+1}}{1-x}$

1. Was it made clear that these techniques can be used in other settings of mathematics?

Estimated Time: 10 minutes

4 Infinite Sums

Put the next two examples on the board and ask the following questions. Put them into Desmos, too.

$$\sum_{1}^{\infty} \frac{1}{2^n} \qquad \text{and} \qquad \sum_{1}^{\infty} \frac{2}{n(n+2)}$$

1. Was it made clear that one needs to find a pattern for s_n based off of a_n?

2. How clear was the connection between a_n and s_n made with these examples?

3. How clear was it that these are two difference sequences?

4. How clear was it that the convergence of s_n is the same as the infinite sum of a_n?

Estimated Time: 20 minutes

Sequences and Series Introduction Worksheet

Name:____

Section: _______

Investigate the sequence $1, 1, 1, \ldots$ and the corresponding series $1 + 1 + 1 + \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

1. What is a general formula for a_n and s_n?

2. What is the relationship between a_n and s_n?

3. Does a_n converge? If so, to what?

4. Does s_n converge? If so, to what?

5. If $\lim_{n\to\infty} a_n \neq 0$, explain how you already know $\lim_{n\to\infty} s_n$ diverges.

Investigate the sequence $a_n = \dfrac{1}{n(n+1)}$ and the corresponding series $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

6. What is a general formula for a_n and s_n?

7. What is the relationship between a_n and s_n?

8. Does a_n converge? If so, to what?

9. Does s_n converge? If so, to what?

10. Does the sum $\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \cdots$ converge or diverge?

11. Does the sum $\displaystyle\sum_{n=1}^{\infty} \frac{b}{n^2 + cn} = \frac{b}{1+c} + \frac{b}{4+2c} + \frac{b}{9+3c} + \cdots$ converge or diverge for any positive integers b and c?

Investigate the sequence $a_n = \left(\frac{1}{5}\right)^n$, $n \geq 0$, and the corresponding series $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

12. Explain why $\frac{1-x^9}{1-x} = 1+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8$. Use this idea for the next few questions.

13. What is a general formula for a_n and s_n?

14. What is the relationship between a_n and s_n?

15. Does a_n converge? If so, to what?

16. Does s_n converge? If so, to what?

17. Find the sum of $a + ar + ar^2 + ar^3 + \cdots$, if $|r| < 1$ by:

a. Finding s_n.

b. Computing $\lim_{n\to\infty} s_n$.

18. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(2)^n}{3^{n+1}} = \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots$.

19. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{5^{n+1}} = \frac{1}{25} - \frac{4}{125} + \frac{16}{625} - \cdots$. Hint: Split up the sequence $a_n = \frac{(-4)^{n-1}}{5^{n+1}}$ into two sequences and find their sums separately.

Sequences and Series Introduction Worksheet

Name:____

Section: _______

Investigate the sequence $1, 1, 1, \ldots$ and the corresponding series $1 + 1 + 1 + \cdots$.

a_1	1
a_2	1
a_3	1
a_4	1
a_5	1
a_6	1
a_n	1

s_1	1
s_2	2
s_3	3
s_4	4
s_5	5
s_6	6
s_n	n

1. What is a general formula for a_n and s_n?
   $a_n = 1$ and $s_n = n$.

2. What is the relationship between a_n and s_n?
   s_n is the sum of the first n terms of a_n.

3. Does a_n converge? If so, to what?
   Yes, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1$.

4. Does s_n converge? If so, to what?
   No, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} n = \infty$.

5. If $\lim_{n\to\infty} a_n \neq 0$, explain how you already know $\lim_{n\to\infty} s_n$ diverges. If $\lim_{n\to\infty} a_n = c \neq 0$ then at some point s_n essentially adds c at every step and no matter how small or large c is this will eventually accumulate to ∞ or $-\infty$. If $\lim_{n\to\infty} a_n$ osciallates like $\lim_{n\to\infty} \sin(n)$ then sn will also oscillate and never converge to a particular value.

Investigate the sequence $a_n = \frac{1}{n(n+1)}$ and the corresponding series $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} \cdots$.

a_1	1/2
a_2	1/6
a_3	1/12
a_4	1/20
a_5	1/30
a_6	1/42
a_n	1/n(n+1)

$s_1$1/2	
s_2	2/3
s_3	7/12
s_4	1/2-1/5
s_5	1/2-1/6
s_6	1/2 - 1/7
s_n	1 - 1/(n+1)

6. What is a general formula for a_n and s_n?
   $a_n = 1/n - 1/(n+1)$ and $s_n = 1/2 - 1/(n+1)$

7. What is the relationship between a_n and s_n?
   s_n is the sum of the first n terms of a_n and a lot of the terms cancel each other out.

8. Does a_n converge? If so, to what?
   Yes, $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

9. Does s_n converge? If so, to what?
   Yes, $\lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$.

10. Does the sum $\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \cdots$ converge or diverge?

    This series diverges since $s_n = \ln(1/2) + \ln(2/3) + \ln(3/4) + \cdots + \ln(n/(n+1)) = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) + \ln(4) + \cdots \ln(n) - \ln(n+1) = -\ln(n+1)$ and $\lim_{n\to\infty} -\ln(n+1) = -\infty$.

11. Does the sum $\sum_{n=1}^{\infty} \frac{b}{n^2+cn} = \frac{b}{1+c} + \frac{b}{4+2c} + \frac{b}{9+3c} + \cdots$ converge or diverge for any positive integers b and c?
    This sum converges since partial fraction decomposition gives $a_n = \frac{b/c}{n} - \frac{b/c}{n+c} = \frac{b}{c}\left(\frac{1}{n} - \frac{1}{n+c}\right)$.
    Then $s_n = \frac{b}{c}\left(\frac{1}{1} - \frac{1}{1+c} + \frac{1}{2} - \frac{1}{2+c} + \frac{1}{3} - \frac{1}{3+c} + \frac{1}{4} - \frac{1}{4+c} + \cdots + \frac{1}{n} - \frac{1}{n+c}\right)$. All of the inside terms are going to cancel out or go to 0 as $n \to \infty$ except for the $\frac{1}{1}$ thus $\lim_{n\to\infty} s_n = \frac{b}{c}$.

Investigate the sequence $a_n = \left(\frac{1}{5}\right)^n$, $n \geq 0$ and the corresponding series $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} \cdots$.

a_1	1
a_2	1/5
a_3	1/25
a_4	1/125
a_5	1/625
a_6	1/3125
a_n	$1/5^n$

s_1	1
s_2	6/5
s_3	31/25
s_4	156/125
s_5	781/625
s_6	3905/3125
s_n	$\frac{1-(1/5)^n}{1-1/5}$

12. Explain why $\frac{1-x^9}{1-x} = 1+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8$. Use this idea for the next few questions.

    Long polynomial division can justify this equation.

13. What is a general formula for a_n and s_n? $a_n = \frac{1}{5^n}$ and using the above equation we get $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ and letting $x = 1/5$ we get $s_n = \frac{1-(1/5)^n}{1-1/5}$.

14. What is the relationship between a_n and s_n?

    s_n is the sum of the first n terms of a_n.

15. Does a_n converge? If so, to what?

    Yes, $\lim_{n\to\infty} (1/5)^n = 0$.

16. Does s_n converge? If so, to what?

    Yes, $\lim_{n\to\infty} \frac{1-(1/5)^n}{1-1/5} = \frac{1}{4/5} = 5/4$.

17. Find the sum of $a + ar + ar^2 + ar^3 + \cdots$, if $|r| < 1$ by:

   a. Finding s_n.
   We use the same idea as problem 13 and get $s_n = a(1 + r + r^2 + \cdots + r^{n-1}) = a\left(\frac{1-r^n}{1-r}\right)$.

   b. Computing $\lim_{n\to\infty} s_n$.
   Here we have to determine when r^n is zero as $n \to \infty$. This happens when $-1 < r < 1$ as noted in the problem statement. Thus $\lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r}$.

18. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(2)^n}{3^{n+1}} = \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots$. In this case we can write $s_n = \frac{1}{3}\left(\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \left(\frac{2}{3}\right)^n\right)$. We have to be careful though when we apply the previous techniques since our other sums started with 1 and we start with 2/3. To fix this we can factor out another 2/3 and get $s_n = \frac{2}{9}\left(1 + \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^{n-1}\right) = \frac{2}{9}\left(\frac{1-(2/3)^n}{1-2/3}\right)$. Now $\lim_{n\to\infty} \frac{2}{9}\left(\frac{1-(2/3)^n}{1-2/3}\right) = \frac{2}{9} \cdot 3 = 2/3$.

19. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{5^{n+1}} = \frac{1}{25} - \frac{4}{125} + \frac{16}{625} - \cdots$. Hint: Split up the sequence $a_n = \frac{(-4)^{n-1}}{5^{n+1}}$ into two sequences and find their sums separately.

   First we will look at the positive parts of the series and call those partial sums $b_n = \frac{1}{25} + \frac{16}{625} + \frac{(-4)^4}{5^6} + \frac{(-4)^6}{5^8} + \cdots = \frac{1}{25}\left(1 + \frac{16}{25} + \left(\frac{16}{25}\right)^2 + \left(\frac{16}{25}\right)^3 + \cdots\right) = \frac{1}{25}\frac{1}{1-16/25} = \frac{1}{25}\frac{25}{9} = 1/9$. Similarly the negative parts of the sum give $c_n = \frac{-4}{125}\left(1 + \frac{16}{25} + \left(\frac{16}{25}\right)^2 + \cdots\right) = \frac{-4}{125}\left(\frac{1}{1-16/25}\right) = \frac{-4}{125} \cdot \frac{25}{9} = \frac{-4}{45}$. Adding the two pieces together we get $b_n + c_n = \frac{1}{45}$.

Sequences and Series for Observer

The first problem is the same as a problem that was discussed in class yesterday. This should tell us if they understand the basics of sequences and series. Ask students to explain number 4 and number 5?

Investigate the sequence $1, 1, 1, \ldots$ and the corresponding series $1 + 1 + 1 + \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

1. What is a general formula for a_n and s_n?
2. What is the relationship between a_n and s_n?
3. Does a_n converge? If so, to what?
4. Does s_n converge? If so, to what?

Student Response(s):

5. If $\lim_{n\to\infty} a_n \neq 0$, explain how you already know $\lim_{n\to\infty} s_n$ diverges.

Student Response(s):

This problem is also similar to a problem done in class. The goal here is to see if students can transfer techniques of integration to a new setting. In particular, partial fraction decomposition coupled with a new idea about telescoping series. The second response box is again about transferring knowledge. Can students use properties of logarithms to see the telescoping series?

Investigate the sequence $a_n = \dfrac{1}{n(n+1)}$ and the corresponding series $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

6. What is a general formula for a_n and s_n?

Student Response(s):

7. What is the relationship between a_n and s_n?

8. Does a_n converge? If so, to what?

9. Does s_n converge? If so, to what?

10. Does the sum $\ln\left(\dfrac{1}{2}\right) + \ln\left(\dfrac{2}{3}\right) + \ln\left(\dfrac{3}{4}\right) + \cdots$ converge or diverge?

Student Response(s):

11. Does the sum $\displaystyle\sum_{n=1}^{\infty} \frac{b}{n^2 + cn} = \frac{b}{1+c} + \frac{b}{4+2c} + \frac{b}{9+3c} + \cdots$ converge or diverge for any positive integers b and c?

Again, a similar problem was discussed in class. This problem is intended to expand ideas about geometric series. Also introduces them to infinite polynomial division and sees if they can generalize a pattern.

Investigate the sequence $a_n = \left(\frac{1}{5}\right)^n$, $n \geq 0$, and the corresponding series $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} \cdots$.

a_1	
a_2	
a_3	
a_4	
a_5	
a_6	
a_n	

s_1	
s_2	
s_3	
s_4	
s_5	
s_6	
s_n	

12. Explain why $\frac{1-x^9}{1-x} = 1 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$. Use this idea for the next few questions.

Student Response(s):

13. What is a general formula for a_n and s_n?
14. What is the relationship between a_n and s_n?
15. Does a_n converge? If so, to what?
16. Does s_n converge? If so, to what?

17. Find the sum of $ar + ar^2 + ar^3 + \cdots$, if $|r| < 1$ by:

a. Finding s_n.

Student Response(s):

b. Computing $\lim_{n \to \infty} s_n$.

Student Response(s):

18. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(2)^n}{3^{n+1}} = \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots$.

19. Extension problem: Find the sum of $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{5^{n+1}} = \frac{1}{25} - \frac{4}{125} + \frac{16}{625} - \cdots$. Hint: Split up the sequence $a_n = \frac{(-4)^{n-1}}{5^{n+1}}$ into two sequences and find their sums separately.